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1 Electric propulsion

An electric motor with propeller, driven by a battery is an excellent power plant for model airplanes. And ability to predict performance before it is built is valuable. This is, to a good approximation, possible for electric propulsion.
The following contains some of the useful formulas for calculating.

1.1 Symbols

Symbols used:

Kv	Motor parameter, rotation speed pr Volt [RPM / Volt].
Kt	Motor parameter, torque pr Ampere [Newton meter / Ampere].
I_0	Motor parameter, Idle current (I-zero) when no load. [Ampere]
Rm	Motor parameter, resistance in motor (with rotation blocked) [Ohm].
Mno	Number of motors driven by battery
Vm	Voltage at motor poles [Volt].
Im	Current through motor [Ampere].
N_0	Idle rotation speed (N-zero) when no load [RPM].
N	Motor rotation speed [Rotations Pr Minute, RPM].
n	Motor rotation speed [Rotations Pr Second, RPS]. n = N / 60.
Qm	Motor torque [Newton meter].
ω	Angular rotation speed [1 / Second]. $\omega = n * 2\pi$
Pin	Input power to motor [Watt]. Pin = Vm * Im
Pout	Mechanical output power delivered by motor [Watt]. Pout = Qm * ω
Pres	Loss due to internal resistance [Watt]. Pres = Im^2 * Rm
Pmec	Loss due to mechanical friction + iron losses [Watt]. Pmec = I0 * Kt * ω
η	Motor efficiency. η = Pout / Pin
Vb	Battery voltage [Volt].
Ib	Battery current [Ampere]. Ib = Im * Mno
Rs	Supply resistance: Battery plus Electronic Speed Controler [Ohm].
Rt	Total resistance, battery plus one motor [Ohm]. Rt = Rs + Rm
Rc	Combined resistance, used when Mno > 1 [Ohm]. Rc = Mno * Rs + Rm
D	Propeller diameter [meter].
A	Propeller pitch (theoretical advance pr. rotation) [meter].
V	Airplane speed [meter / second].
np	Propeller rotation speed [RPS].

J	Dimensionless advance ratio. J = V / (np * D)
Cp	Dimensionless power coefficient for propeller. Is a function of J.
Ct	Dimensionless thrust coefficient for propeller. Is a function of J.
ρ	Air density [kg / Meter^3]. At zero altitude ρ = 1.225 kg/m^3.
PpIn	Input power to propeller [Watt]. PpIn = Pout
PpOut	Power output from propeller as thrust times speed [Watt].
ηProp	Propeller efficiency. η = PpOut / PpIn

2 Electric motors

Electric motors having permanent magnets, both with brush and brushless, can be described by a relatively simple model, which turns out to be a good approximation to a "real" motor. The model provides a set of formulas, based on describing parameters. The parameters are motor specific.

2.1 *Electric motor parameters*

The motor is characterized by three constant parameters: I_0, Rm and Kv.

The mechanical and hysteresis losses are assumed to be independent of voltage and current (but increasing linear with rotation speed) and are characterized by I_0, the idle current. The torque used to turn the motor when idle may be calculated from I_0 ($Q_0 = I_0 * Kt$) and this torque is assumed to be independent of rotational speed. This approximation seems to be fine for most motors.

Two fundamental equations describe the motor:
The connection between current, voltage and rotation speed is

$$Im = (Vm - N / Kv) / Rm$$

=> $$N = Kv * (Vm - Im * Rm)$$

The torque developed on the motor axis is

$$Qm = (Im - I_0) * Kt$$

The motor parameters may be given in other forms, e.g. current at a given voltage with rotation blocked (named stall current), which allows you to derive Rm = Vm / current. Or idle rotation speed N_0 at a given voltage, which allows you to derive Kv $= N_0 / (Vm - I_0 * Rm)$. In stead of Kv the parameter Kt may be given, from which Kv may be calculated.

2.2 Motor efficiency

The power output, losses and power input are connected

$$Pout = Pin – Pres – Pmec$$

the input power is

$$Pin = Im * Vm$$

the loss due to ohmic resistance (Cu loss) is

$$Pres = Im\^2 * Rm$$

the loss due to mechanical friction and hystresis (Fe loss) is friction torque times the rotation speed. Friction torque is I_0 * Kt (easy to se/understand when you remember that external torque Qm = 0 = ($I_0 – I_0$) * Kt

$$Pmec = I_0 * Kt * \omega$$

From this we get

$$Pout = Im * Vm – Im\^2 * Rm – I_0 * Kt * \omega$$

From $Pout = Qm * \omega$

$= Pin – Pres – Pmec$

and $\omega = N * 2\pi / 60 = Kv * (Vm – Im * Rm) * 2\pi / 60$

we can find the proportionality between Kv and Kt

$$Qm * \omega = (Im – I_0) * Kt * \omega$$

$$= Vm * Im – Im\^2 * Rm – I_0 * Kt * \omega$$

=> $Im * Kt * \omega = Im * Kt * Kv * (Vm – Im * Rm) * 2\pi / 60$

$= Vm * Im – Im\^2 * Rm$

=> $Kt = 60 / (Kv * 2\pi)$

From this we get

$$Kt * \omega = (60 / (Kv * 2\pi)) * N * 2\pi / 60$$

$$= N / Kv = Kv * (Vm – Im * Rm) / Kv$$

$$= Vm – Im * Rm$$

The motor efficiency η is

$$\eta = Pout / Pin$$

after a number of transformations and reductions we get

$$\eta = (1 – I_0 / Im) * (1 – Im * Rm / Vm)$$

If we assume that Vm does *not* depend on Im, we can find maximum for η(Im) by differentiation and setting the first derivative equal to zero.

$$d \eta / d Im = (Im\^2 * Rm – I_0 * Vm) / Im\^2 * Vm$$

=> $d\eta / d\,Im = 0$ for $Im = sqrt(Vm * I_0 / Rm)$

$Im = sqrt(Vm * I_0 / Rm)$ turns out to be the maximum requested

2.3 Battery and motor

When a real battery and motor controller is used, then Vm *is* dependant on Im (unless Rs = 0, which is unphysical). Thus maximum η is found for slightly different value of Im. A good approximation to maximum η can be found by iteration, using $Vm_n = Vb - Im_{n-1} * Rs$ and then $Im_n = sqrt(Vm_n * I_0 / Rm)$.

For systems with only one motor (Mno = 1) we get:

Ib = Im

Vm = Vb – Ib * Rs = Vb – Im * Rs

Total resistance as seen by the battery is

Rt = Rs + Rm

Inserting Vm as function of Vb in the fundamental equation gives

N = Kv * (Vm – Im * Rm) = Kv * (Vb – Im * Rs – Im * Rm)

= Kv * (Vb – Im * Rt)

=> Im = (Vb – N / Kv) / Rt

Inserting Vm as function of Vb in the expression for Pout we get (after reductions are made)

$Pout(Im) = Vm * Im - Im\text{^}2 * Rm - I_0 * Kt * \omega$

$= Im * (I_0 * Rt + Vb) - Im\text{^}2 * Rt - I_0 * Vb$

$= (Im - I_0) * (Vb - Im * Rt)$

We may also express Pout as a function of N

$Pout(N) = Vm * Im - Im\text{^}2 * Rm - I_0 * N / Kv$

after a number of transformations and reductions we get

$Pout(N) = N * (N_0 - N) / (Kv\text{^}2 * Rt)$

If we want to see if the motor can provide a given power E, we first solve the second order equation (calling the unknown current Ix)

$E = Pout(Ix) = Ix * (I_0 * Rt + Vb) - Ix\text{^}2 * Rt - I_0 * Vb$

=> $Ix\text{^}2 * Rt - Ix * (I_0 * Rt + Vb) + I_0 * Vb + E = 0$

This is solved to give

$$Ix = (I0 * Rt + Vb \pm sqrt((I0 * Rt + Vb)^2 – 4 * Rt * (I0 * Vb + E))) / (2 * Rt)$$

Of a physical solution we must demand that it is real (Determinant >= 0) and that $I_0 < Ix < Vb / Rt$

2.4 More than one motor

We assume that the motors are equal. If more than one motor draws current from one speed controller and battery (Mno > 1) , we get

$$Ib = Mno * Im$$

$$Vm = Vb – Ib * Rs = Vb – Im * Mno * Rs$$

Total resistance as seen by the battery is

$$Rt = Rs + Rm / Mno$$

We define a symbol for a combined resistance, which is used below

$$Rc = Mno * Rs + Rm = Rt * Mno$$

For *each* motor we have

$$N = Kv * (Vm – Im * Rm) = Kv * (Vb – Mno * Im * Rs – Im * Rm)$$
$$= Kv * (Vb – Im * Rc)$$

$$N_0 = Kv * (Vb – I_0 * Rc)$$

$$Im = (Vb – N / Kv) / Rc$$

$$Pin = Vm * Im = (Vb – Im * Mno * Rs) * Im$$

$$Pout(N) = Vm * Im – Im^2 * Rm – I_0 * N / Kv$$
$$= Im * Vb – Im^2 * Rc – I_0 * N / Kv$$
$$= N * (N_0 – N) / (Kv^2 * Rc)$$

3 Propeller

When the power output from the motor is known, it is possible to make an estimate of the static and dynamic thrust given by a propeller, provided the diameter and the two dimensionless coefficients Cp(J) and Ct(J) are known. The two coefficient functions vary with J and are specific for each propeller type. They must be found in a wind tunnel or calculated from propeller geometry.
We mount a propeller on each motor.

3.1 Power and thrust

We assume that Cp(J) and Ct(J) are known. We can then calculate the power needed to turn the propeller, and the thrust generated for any value of the dimensionless advance parameter J, propeller rotation speed np [RPS] and propeller diameter D [meter].

Power absorbed by the propeller:

PpIn = Cp(J) * ρ * np^3 * D^5 [Watt]

Thrust generated:

Thrust = Ct(J) * ρ * np^2 * D^4 [Newton]

3.2 Rotation speed

For a given value of J, we may find a solution to the rotation speed of the motor and propeller. The power produced by the motor must be equal to the power absorbed by the propeller, and this determines the rotation speed. Useful values for J range from 0 to the relative pitch value (Pitch / Diameter). The motor power output curve is a parable function of N (going as - N^2), and the power absorbed by the propeller increases (going as np^3) so the two curves will intersect for some value of N and np. Propeller rotation speed np will be different from N, if the propeller is driven via a gear.

np = N / (60 * gearRatio) [RPS]

Setting the two power values equal results in a third degree equation.

Pout(N) = PpIn(J, np)

=> N * (N_0 – N) / (Kv^2 * Rc) = Cp(J) * ρ * (N / (60 * gearRatio))^3 * D^5

This third degree equation in N immediately reduces to second degree (as solution N = 0 is not interesting). The second degree equation is easily solved; only the positive solution for N is physical.

3.3 Propeller efficiency

From the calculated np, and the given J we get the speed V = J * np * D [meter / second].

Now PpOut can be calculated.

PpOut = Thrust(J, np) * V [Watt]

and from that we can get the propeller efficiency

ηProp = PpOut / PpIn

4 Propeller Cp and Ct

Each propeller type has its own sets of Cp(J) and Ct(J) coefficients, dependant on blade area, section pitch and profiles used et c. But Cp and Ct tabulations for a given model propeller are normally not available, and it takes careful experiments in a wind tunnel to obtain them.

4.1 *Rough Cp and Ct estimates*

The second best solution, when Cp and Ct are not known and no wind tunnel is at hand, is to take a set of Cp and Ct values for an existing family of propellers, and use them as rough guideline for the propeller we are interested in. I have used the propeller family published in N.A.C.A. Technical report 640 as a basis.
The family described in TR 640 was obtained by varying the pitch of the given blades (simply turning them in their holder), and varying the number of blades.
This gives a set of measured curves for Cp(J, Pitch, BladeNo) and Ct(J, Pitch, BladeNo).

My idea is to interpolate/extrapolate between the TR 640 curves, using the model propellers relative pitch and number of blades as controlling parameters. I approximate the curves in the TR640 family with broken lines, and interpolate between the lines, as can be seen on the two figures below.

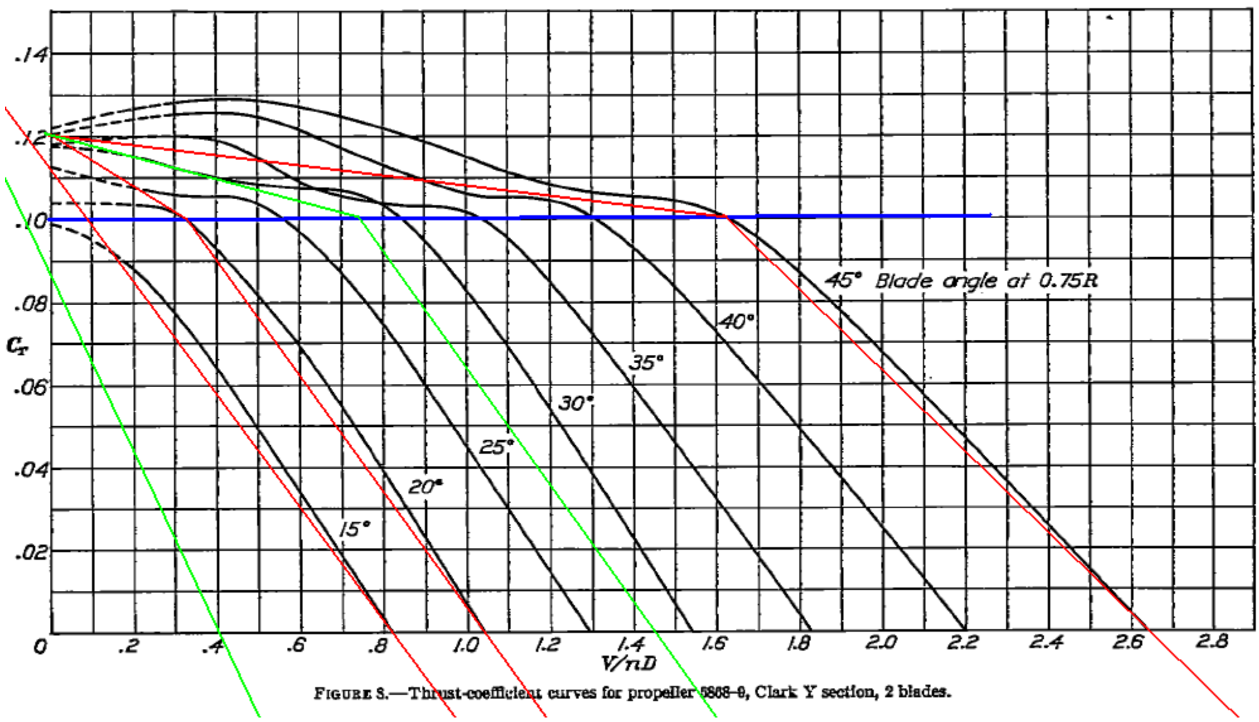

FIGURE 8.—Thrust-coefficient curves for propeller 5868-9, Clark Y section, 2 blades.

Ct curves and interpolation lines.

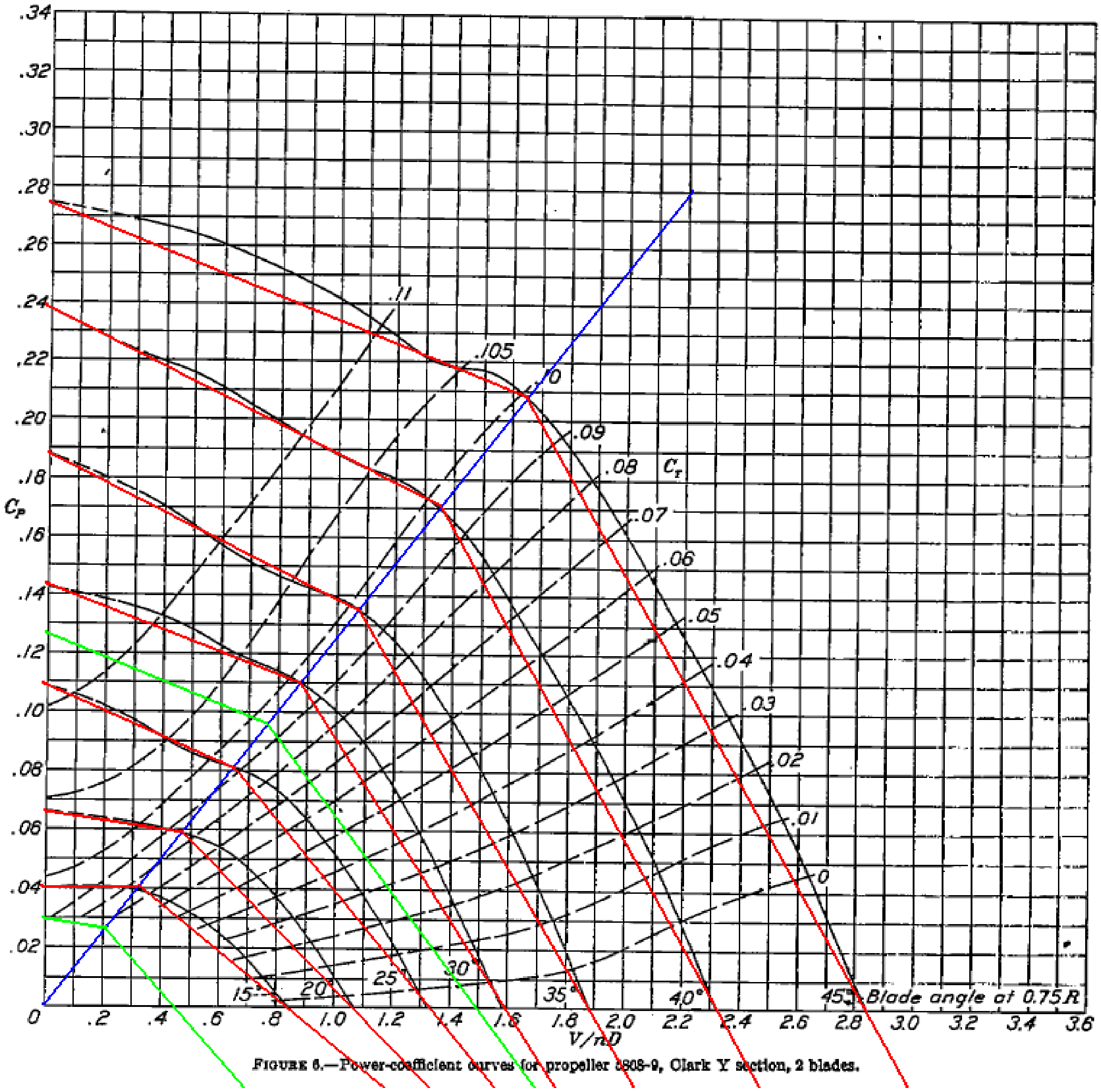

FIGURE 6.—Power-coefficient curves for propeller 5868-9, Clark Y section, 2 blades.

Cp curves and interpolation lines.

The figures show:

- black curves, which are the measured values in TR 640

red lines, which exemplifies my approximation to the given family curves
blue lines which are helper lines
green lines, which are interpolations made for a two example pitch values.
The lines are not drawn with high precision; they are only meant to show the principle. The broken "red" and "green" lines calculated in the program must be

consulted for more precise values.

Cp and Ct are proportional to number of blades, except for an interference factor. Values for more than two blades are approximated from two blade propeller data by multiplication by (factor^(noBlades – 2)), where factor is 0.93. This attempts to take care of the fact, that the wake created by a blade changes the conditions for the following blades, and reduces the thrust generated, but also reduces the power needed to turn the propeller somewhat.

4.2 Formulas using Cp and Ct estimates

With the estimated Cp(J) and Ct(J) functions (“green lines”) we can now calculate the thrust and power. The formulas are repeated here:

Power absorbed by the propeller:

$$PpIn = Cp(J) * \rho * np^3 * D^5 \quad [Watt]$$

Thrust generated:

$$Thrust = Ct(J) * \rho * np^2 * D^4 \quad [Newton]$$

Propeller efficiency:

$$\eta Prop = PpOut / PpIn = T * V / P = J * Cp(J) / Ct(J)$$

The propeller efficiency ηProp will never exceed 90% in all practical cases.
This condition can then be used to make a “sanity check” on the Ct value used, if we thrust the Cp value.

4.3 Traditional power and thrust estimates

Searching WWW gives (direct or indirect) two formulas for static power and thrust, which seem to be used by many model flyers and manufacturers.
I have only found static formulas (Cp(0) and Ct(0)), but not found any estimates for nonzero values of J.

The formulas quoted below are taken from Chuck Gads JavaScript program. In his scheme, an effective diameter is used as a method to take care of more than two blades (and the interference this gives). All lengths are in inches. The number PConst is dimensionless, close to 1, and is used to adapt to different propeller types.

effdiameter = propdiameter * (noBlades / 2)^0.2 [inches]

thrust = proppitch * effdiameter^3 * (proprpm / 1000)^2 * 28.3 * 0.981 / 10000

[grams]

propwatts = PConst * (proppitch / 12) * (effdiameter/12)^4 * (proprpm / 1000)^3 [Watt]

If we want to reformulate, to the usual thrust and power formulas, we must find values for EquivCt(0) and EquivCp(0), derived from the web approximation. We define a relative pitch value RelPitch = A/D = proppitch / propdiameter .
Setting the formulas from the Web equal to the ones which use Cp and Ct now can be solved to give equivalent values for Cp(0) and Ct(0):

EquivCp(0) = RelPitch * 0.0670
EquivCt(0) = RelPitch * 0.210

The difference between various types of propellers is taken care of by the simple multiplier "PConst" to Cp, with values near 1.0.

EquivCp(0) = RelPitch * 0.0670 * PConst

I have not seen any formulas with multipliers for Ct, but the idea is implied as a "TConst" column in the list of propeller data given at WWW.MotoCalc.com . I define

EquivCt(0) = RelPitch * 0.210 * TConst

setting TConst = 1.0 when no better information is available.

The idea of using an interference factor (= 0.93 in the new scheme) for propellers with more than two blades, is nearly equivalent to the use of effdiameter when calculating absorbed power. Thrust uses effdiameter^3, and the values agree less.

	(NoBlade/2)* factor^(NoBlades – 2)	effdiameter^4 / propdiameter ^4
2	1.0	1.0
3	1.395	1.383 (Thrust: 1.275)
4	1.729	1.741 (Thrust: 1.515)
5	2.011	2.081 (Thrust: 1.732)

Values of EquivCt(J) for J > 0 are approximated by a line from (0.0, EquivCt(0)) to (RelPitch, 0.0).

Similar EquivCp(J) values for J > 0 are approximated by a line from (0.0, EquivCp(0)) to (RelPitch * 1.05, 0.0). The reason for the factor 1.05 is that Cp must be positive for J = RelPitch; the propeller needs power to turn it, even when no thrust is generated.

4.4 Comparison old - new

The “green line” Cp(0) and Ct(0) coefficients derived from NACA TR 640 by interpolation, are constantly smaller than the values obtained by the Web formulas, even when the PConst factor is set to 1.0.

I take that this is connected to the blade area. Model propellers looks like having bigger blade area than the blades used in TR 640. When I make a 20% increase of the blade area, then Cp(0) and Ct(0) as found in the new scheme, gets reasonably close to the EquivCp(0) and EquivCt(0) calculated from the WWW formulas, in the most important range of RelPitch (0.5 to 1.2).

It is clear that the Web estimates are problematic for both low and high values of RelPitch. The EquivCp(0) estimate, where Cp is proportional to RelPitch , cannot be correct for very low RelPitch values, as the propeller absorbs power even when no thrust is produced.

And the Web EquivCt(0) estimate, where Ct is proportional to RelPitch, does increase without limit for high values of RelPitch, but any real high pitch propeller will get into a stalled blade condition, which limits the static thrust the propeller can make.
Also the value of propeller efficiency ηProp calculated from the Web EquivCp(J) and EquivCt(J) are often higher than 90%, in fact it may even become more than 100%. This is unphysical, and shows the weakness of the approximations.

4.5 Suggested Cp and Ct values

I have chosen to use the “green line” Cp(J) and Ct(J) coefficients, with the factor 0.93 to take care of more than 2 blades. This is, I guess, a fair approximation to real propellers, and it also provides us with values for $J > 0$.
But I also chose to keep the PConst and TConst multipliers. The reason for keeping the multipliers is, that it probably is the closes we get to wind tunnel data for model propellers, and I assume that the multipliers do represent some real knowledge from experiments.

The “green line” Cp(J) and Ct(J) coefficients leads to reasonable ηProp in nearly all cases. The few exceptions are “cured” by reducing Ct(J) to a value consistent with max 90% propeller efficiency, which is physically reasonable.

5 JavaScript program

Based on a JavaScript program made by Chuck Gad in U.S.A. (found on WWW) I have made a program following the formulas given above. The file was fetched from Chuck Gadd's homepage 2006-08-08 (link: http://www.csd.net/~cgadd/eflight/calcs_motortest.htm), and modified and extended.

The JavaScript program uses the new Cp and Ct values.

Please note: Even if it is not better, I have learned a lot, and it has been great fun to make it !

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